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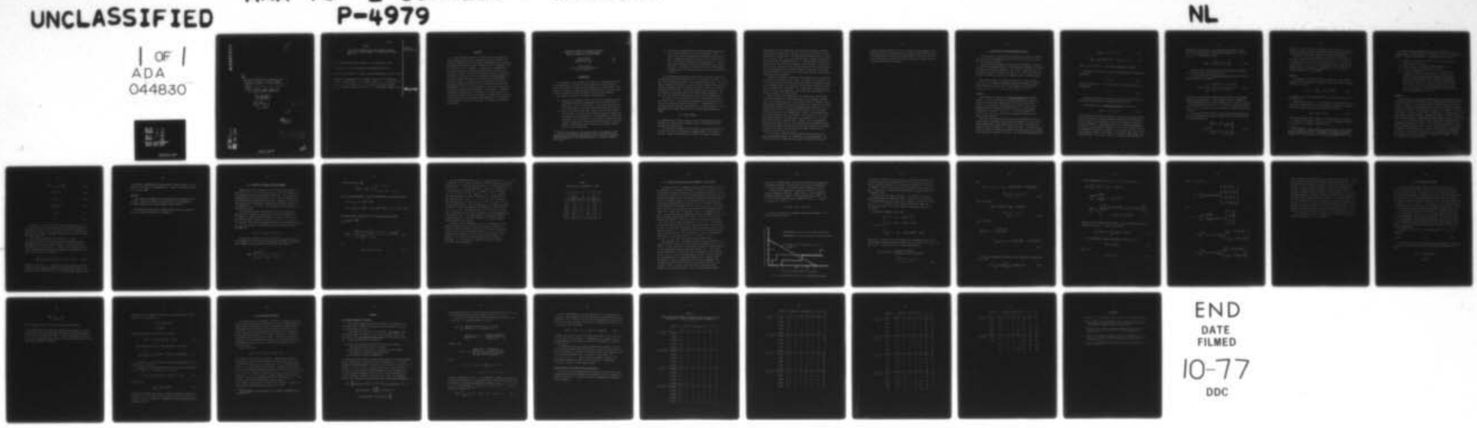

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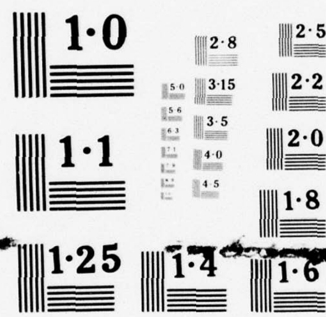
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OPTIMAL DISPATCHING OF AN INFINITE CAPACITY SHUTTLE: CONTROL AT A SINGLE TERMINAL

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Edward Ignall ~~and~~ Peter Kolesar

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In the first expression for C on page 21, i t/2 should be $(i-1)t/2$.

In consequence, i/2 in the second expression should be $(i-1)/2$.

Therefore, the calculations showing β should be larger than λ_2 are in error.

Stopping-rule formulations of the problem of section III, developed subsequently, suggest that $\beta = \lambda_2$ is optimal. Using $\beta = \lambda_2$ in the correct expression for C gives queue sizes 5 to 15% higher than those resulting when both queue sizes are known, not the 25 to 75% reported on pages 3 and 16.

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ABSTRACT

We study the optimal control of a shuttle system consisting of a single infinite capacity carrier transporting passengers between two terminals is studied. Passengers arrive according to independent Poisson processes, and dispatching decisions to hold the carrier for more passengers can be made at only one of the terminals. The objective is minimization of the long-run average of a linear passenger waiting cost and a fixed charge per trip made. When complete information about the system state is available, and travel times are deterministic, ~~we prove that~~ it is optimal to dispatch the carrier if, and only if, the total number of passengers waiting at both terminals is greater than a cutoff value. An iterative method for computation of the cutoff value is given and ~~we find~~ it is found that it can be well approximated by a function of system costs and parameters similar to the economic lot size formula. A (possibly non-optimal) dispatching rule is proposed for the case when only the number of passengers waiting at one terminal is known, and its efficiency is compared to that of the aforementioned optimal rule. Extensions to other optimality criteria and to the case of stochastic travel times are outlined.

OPTIMAL DISPATCHING OF AN INFINITE CAPACITY SHUTTLE: CONTROL AT A SINGLE TERMINAL

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INTRODUCTION

In this paper we study a mathematical model of a transportation system in which passengers (or goods) are transported between two terminals by a single carrier with very large capacity compared to the arrival rate of passengers. Our goal is to determine optimal operating rules for such systems in which dispatching decisions are made at only one of the terminals.

The system we study has the following characteristics:

(1) There are two terminals labeled 1 and 2 at which passengers arrive individually according to independent Poisson processes with arrival rates λ_1 and λ_2, respectively. All arriving passengers wait to be transported to the other terminal where they exit the system.

(2) There is a single carrier which shuttles back and forth between the terminals. The carrier has an infinite capacity, meaning that on departure from the terminal it can accommodate all the waiting passengers. For the majority of our analysis, we assume that inter-terminal travel time is deterministic and independent of the load carried. Some analysis when travel times are random variables will be found in Section IV.

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(3) The shuttle is operated with the following dispatching mechanism: At one of the terminals, the carrier never waits. Rather, it visits this station, instantaneously picking up those passengers who are already there--much as a bus does at the typical urban bus stop. At the other terminal, the carrier can wait. In fact, the only control possible is exerted by deciding how long it should wait there. Our concern in this paper is with the determination of optimal rules for dispatching the carrier at the terminal where it can wait.

We use minimum long-run average cost per unit time as our standard of optimality and consider two kinds of costs: a fixed cost of making a trip, K, and a cost of passenger waiting, h, per passenger per unit time. Our principal result, developed in Section I, is that, when the dispatcher has available complete information about the state of the system, there is an optimal dispatching rule of the form: The carrier should depart if, and only if, the total number of customers waiting in the system exceeds a constant. Our main tool is a result of Deb and Serfozo [2] for a one terminal system.

We refer to such a rule as a control limit rule and to the constant as a control limit. An expression for average cost as a function of the control limit constant is given in Section II. Computations with this expression indicate that the optimal value of the control limit, k*, is well approximated by a simple function of the cost and arrival parameters, namely

$$k^* = \sqrt{2(\lambda_1 + \lambda_2)K/h} \ .$$

Our computations also indicate that, if one can choose which terminal to dispatch from, average cost is slightly lower if the terminal with more passenger arrivals is chosen.

In Section III we consider the case where the dispatcher does not know how many passengers are waiting at the <u>other</u> terminal. The results of Section I suggest the desirability of a rule of the form: The carrier

should depart from the terminal if, and only if, the number of customers waiting at the terminal plus a multiple of the time the carrier has waited there exceeds a control limit. An expression for average cost as a function of the multiple, call it β, and the control limit is obtained, and calculations reveal that the optimal value of β is typically larger than the passenger arrival rate at the other terminal. More important is the observation that, for a fixed trip rate, the dispatching rule using only passenger waiting information from one terminal results in 25 percent to 75 percent higher average queue sizes than the optimal dispatching rule using complete passenger waiting information.

Section IV is devoted to stochastic travel times where it is indicated that the result of Section I still holds. In Section V, an alternative way of viewing the problem--minimizing the average number of passengers waiting at the terminals subject to a constraint on the average number of trips per hour by the shuttle--is considered. It is shown that, for properly chosen values of the trip rate constraint, rules of the form shown optimal for the problem of Section I are again optimal.

One may question the relevance of these results since real shuttles have finite capacity and it would seem that, for such systems, one ought then to pay attention to i_1 and i_2 (the number waiting at terminals 1 and 2, respectively) individually and not just to their sum. However, the infinite capacity model would be useful if the following conjectures were true. (We think they are.) First: An optimal infinite capacity rule is an upper bound for an optimal finite capacity rule. That is, if an optimal rule in the infinite capacity case is to dispatch if, and only if, $i_1 + i_2 \geq k^*$, then an optimal rule in the finite case is to dispatch if $i_1 + i_2 \geq k^*$, and possibly in other situations also. Second: If k^* is less than the capacity of the shuttle and if, in the infinite case, the probability that the shuttle has room for all who are waiting when it arrives at a terminal is close to one, then an optimal rule for the finite case is close to: dispatch if, and only if, $i_1 + i_2 \geq k^*$.

There has been relatively little reported work on shuttles or on other multiterminal systems with probabilistic passenger arrivals. Our previous work [3] was restricted to the case of a shuttle which could

transport only one passenger at a time. Barnett [1] has compared average number waiting for several operating rules for an infinite capacity shuttle. His approach differs from ours in several respects: Ours has a cost per trip and allows travel time to be stochastic, while his allows the shuttle to wait at either of the terminals.

I. THE FORM OF AN OPTIMAL DISPATCHING POLICY

For definiteness, suppose that the shuttle may wait only at terminal 1. Let λ_1 and λ_2 be the passenger arrival rates, and r the round trip travel time (r/2 in each direction).

Decisions are permitted upon the arrival of the shuttle at terminal 1 and upon every passenger arrival--at either terminal--thereafter, as long as the shuttle is waiting at terminal 1. The cost of a round trip is $K \geq 0$. Passenger waiting cost is linear with rate $h \geq 0$ so that, when there are a total of i passengers waiting at the terminals, cost is incurred at rate $h \cdot i$.

The state at epoch t is given by the pair $\underline{i}(t) = (i_1(t), i_2(t))$, where $i_k(t)$ is the number of passengers waiting at terminal k at epoch t. Wherever clarity will not be lost, we will suppress the t and write i_1, i_2, etc. Let $\lambda = \lambda_1 + \lambda_2$, $\delta_1 = (1, 0)$, and $\delta_2 = (0, 1)$.

Problem 1

We will restrict ourselves to stationary Markovian (SM) control policies: that is, those where the decision made when the system is in state $\underline{i}$ at a review epoch depends only on the state at that epoch. A SM rule can be viewed as a specification, for each (i_1, i_2), of the probability that we dispatch the shuttle when (i_1, i_2) are waiting. Consequently, if we let $I = \{0, 1, 2, ...\}$, then any function $A(.,.)$ from $I \times I$ onto $[0, 1]$ is a SM rule for our system, and any SM rule induces such a function.

Under any such operating rule, our system, when observed at decision epochs, forms a semi-Markov process whose key elements are a set of transition probabilities, sojourn times and one step costs. We follow the notation of Ross [5, Chapter 7]. There are 2 actions: the carrier can be held until the next decision epoch, denoted by W (for wait); or the carrier can be dispatched, denoted by G (for go). The transition probabilities are:

$$p^W_{\underline{i},\underline{i}+\delta_k} = \lambda_k/\lambda \qquad k = 1, 2 \tag{1.1}$$

$$p^G_{\underline{i},\underline{m}} = a_{m_1}(\lambda_1 r)a_{m_2}(\lambda_2 r/2) \qquad m_1, m_2 = 0, 1, \ldots \tag{1.2}$$

where $a_j(z) = e^{-z}z^j/j!$ is the Poisson probability function.

The sojourn or transition times are exponential random variables with expectation

$$\tau^W = 1/\lambda, \tag{1.3}$$

when the shuttle waits. If the shuttle goes, the transition time is a constant and

$$\tau^G = r. \tag{1.4}$$

We shall be interested in long-run average cost per unit time and so must concern ourselves with the expected transition costs:

If the shuttle waits, <u>expected transition cost</u> is given by

$$C^W(\underline{i}) = h \cdot (i_1 + i_2)/\lambda, \tag{1.5}$$

since the number of passengers waiting is constant over the interval until the next one arrives. To obtain expected transition cost if the shuttle goes, we define the random variable $I_k(t|\underline{i})$ as the number of passengers waiting at terminal k t minutes after the shuttle departs from terminal 1, if the state was $\underline{i}$ just before departure, $(t \leq r)$. When the shuttle leaves, all passengers now waiting at terminal 1 are picked up, and, after r/2, all then waiting at terminal 2 are picked up. Additions to both queues are by Poisson arrivals. So we have: $I_1(t|\underline{i})$ is a Poisson random variable with

parameter $\lambda_1 t$, and $I_2(t|\underline{i}) - i_2$ is Poisson with parameter $\lambda_2 t$ for $0 \leq t \leq r/2$, while $I_2(t|\underline{i})$ is Poisson with parameter $\lambda_2(t - r/2)$ for $r/2 \leq t \leq r$. Consequently, we have

$$C^G(\underline{i}) = K + \frac{hr^2}{2}\left(\lambda_1 + \frac{\lambda_2}{2} + \frac{i_2}{r}\right). \tag{1.6}$$

We wish to find a dispatching policy that minimizes the average cost per unit time of operating this system.

It is well known that the <u>average cost per unit time</u> of operating under control rule A is given by

$$g_1(A) = \frac{\sum_{\underline{i}} \pi_{\underline{i}} \cdot (A(\underline{i})C^G(\underline{i}) + [1 - A(\underline{i})]C^W(\underline{i}))}{\sum_{\underline{i}} \pi_{\underline{i}} \cdot (A(\underline{i})\tau^G + [1 - A(\underline{i})]\tau^W)} \tag{1.7}$$

where $\{\pi_{\underline{i}}\}$ is the stationary probability distribution of the imbedded Markov chain under policy A. (The subscript 1 refers to Problem 1.) If these probabilities do not exist (implying that the chain is transient or null recurrent), we take $g_1(A) = \infty$. Of course, for any policy with $A(\underline{i}) = 1$ for $i_1 \geq i_1{}^*$ and $i_2 \geq i_2{}^*$, the process is ergodic and $g_1(A)$ will be finite.

It is also well known (Ross [5]) that a policy which minimizes g_1 results from choosing the minimizing action for each $\underline{i}$ which satisfies the functional equation:

$$v_{\underline{i}} = \min \left\{ \begin{array}{l} \left(C^W(\underline{i}) - g\tau^W + \sum_{\underline{j}} p^W_{\underline{i}\underline{j}} v_{\underline{j}}\right) \\ \\ \left(C^G(\underline{i}) - g\tau^G + \sum_{\underline{j}} p^G_{\underline{i}\underline{j}} v_{\underline{j}}\right) \end{array} \right\} \tag{1.8}$$

However, (1.8) is inconvenient in several ways--it does not reveal the form of the optimal policy and computations cannot be made directly with it since there are an infinite number of equations and variables.

In order to show that an optimal SM policy exists for Problem 1 of the form: Dispatch the shuttle if (and only if) $i_1 + i_2 \geq k$, and also to determine a value of k which minimizes average cost, we consider two related problems. The first of these problems is used to show that the optimal policy need depend only on knowledge of $i_1 + i_2$ and does not require knowing i_1 and i_2 individually.

Problem 2

This problem is the same as Problem 1 except that we also charge a waiting cost for passengers from terminal 1 during their trip to terminal 2. Formally, we then have

$$C^G(\underline{i}) = K + \frac{hr^2}{2}\left(\lambda_1 + \frac{\lambda_2}{2} + \frac{i_1 + i_2}{r}\right) \qquad (1.9)$$

and Problem 2 is to minimize average cost for the system given by (1.1)-(1.5) and (1.9).

Consider any SM policy A. It is elementary to prove that the long-run average cost obtained when this rule is applied to system 2 is related to the long-run average cost when it is applied to system 1 by

$$g_2(A) = g_1(A) + h\lambda_1 r/2, \qquad (1.10)$$

because every passenger from terminal 1 eventually travels to terminal 2.*

Equation (1.10) implies: any SM policy that is optimal for Problem 1 is also optimal for Problem 2, and vice versa.

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*More formally: over an interval t units long, $\lambda_1 t + e(t)$ passengers arrive at terminal 1 and travel to terminal 2, incurring total "extra" waiting cost $hr/2(\lambda_1 t + e(t))$. The average of this cost per unit time is then $hr/2(\lambda_1 + e(t)/t)$, and by the law of large numbers $e(t)/t \to 0$ as $t \to \infty$.

We now argue that an optimal control rule need depend only on $i_1 + i_2$. A formal argument, relating Problem 2 to Problem 3 (which will be introduced shortly), is given in the Appendix. An informal one goes as follows:

(1) $C^G(\underline{i})$ can be written in the form $K' + b(i_1 + i_2)$ and hence is a function only of $i_1 + i_2$.

(2) $C^W(\underline{i}) = h/\lambda(i_1 + i_2)$, again a function only of $i_1 + i_2$.

(3) τ^W and τ^G and $p^G_{\underline{i},\underline{m}}$ do not depend on $\underline{i}$ at all.

(4) The $p^W_{\underline{i},\underline{m}}$ are such that $m_1 + m_2 = i_1 + i_2 + 1$; that is, if decision W is made, the system evolves to a new state with exactly one more customer waiting (somewhere) in the system. It has been observed above that the expected cost $C^W(\underline{i})$ depended only on the sum $i_1 + i_2$, and the costs $C^W(\underline{m})$ or $C^G(\underline{m})$, one of which is now to be incurred, depend only on the sum $i_1 + i_2 + 1$ so that all future evolution and costs of concern to us are dependent only on $i_1 + i_2$.

Problem 3

Deb and Serfozo [2] consider an infinite capacity batch service queueing system with a Poisson arrival process and a similar cost and decision structure to that of Problem 1. They show that long-run average cost per unit time is minimized by a policy of beginning a service only if the number of customers in queue is at least k*. Their problem is similar to our Problem 1, but their system takes into account only activity at terminal 1. We will show how we can relate their problem to Problem 2, and hence to Problem 1. First, we describe their problem: Consider a single terminal (Deb-Serfozo) system with arrival rate $\lambda_1 + \lambda_2 = \lambda$ at the single terminal. Suppose that round trip travel time is deterministic and equal to r, that B + dj is the cost of a round trip made with j passengers on board, and $h > 0$ is the waiting cost rate. Let i denote the number of waiting passengers. Decisions to wait or go are permitted at the epochs of the arrival of the carrier at the terminal and at passenger arrival epochs as long as the carrier is waiting at the terminal. This is a semi-Markov decision process with:

$$C^G(i) = B + di + \frac{hr^2}{2}\lambda \tag{1.11}$$

$$C^W(i) = hi/\lambda \tag{1.12}$$

$$P^W_{i,i+1} = 1 \tag{1.13}$$

$$P^G_{ij} = a_j(\lambda r) \tag{1.14}$$

$$\tau^W = 1/\lambda \tag{1.3}$$

$$\tau^G = r \tag{1.4}$$

As before, we wish to minimize the long-run average cost per unit time for this system, and Deb and Serfozo have proved that, among the optimal rules, there is a rule such that the carrier is dispatched if, and only if, the number of passengers waiting in line exceeds a minimum level called, say, k^*.

Now, recall our assertion that for Problem 2 we need only concern ourselves with the total number of passengers waiting. That means that, for the 2-terminal system, we need only observe the total number in the system, say, i. Problem 2 when viewed in this way forms a semi-Markov process whose key parameters are defined by (1.3), (1.4), (1.9), (1.12), (1.13) and by

$$P^G_{ij} = \sum_{k=0}^{j} a_k(\lambda_1 r)\, a_{j-k}(\lambda_2 r/2) = b_j(\lambda_1, \lambda_2, r). \tag{1.15}$$

Letting $d = hr/2$ and $B = K - \lambda_2 hr^2/4$ yields equality between (1.9) and (1.11) so the only remaining difference between Problems 2 and 3 is the difference between (1.14) and (1.15). But the particular form of (1.14)

is incidental. Examination of Deb and Serfozo's proofs reveals it is sufficient for the p_{ij}^{G} to be independent of i. Since this is also the case for (1.15) we claim:

Theorem 1

There exists an optimal (long-run average cost) dispatching rule of the form: Dispatch the carrier if, and only if, the total number of passengers waiting in the system exceeds a control limit k*.

Note that this result can be extended to more than two terminals as long as the shuttle is held only at terminal 1.

II. FINDING AN OPTIMAL DISPATCHING NUMBER

Having shown that an optimal SM rule is of the form: Dispatch if, and only if, $i_1 + i_2 \geq k$, we present a method by which the optimal k can be determined given the system parameters and costs. We provide below an expression for g, the average cost per unit time, as a function of k which can easily be evaluated with the aid of a computer. We have made calculations with this expression showing how the optimal k depends on the parameters $(\lambda_1, \lambda_2, r, h, K)$. These results will be discussed at the end of this section.

We do our calculations using Problem 3, and then, using (1.10), subtract $h\lambda_1 r/2$ from the resulting value of g to get the average cost for Problem 1. Following Deb and Serfozo, let T_n be the epoch of the nth shuttle arrival at terminal 1, and $T_0 \equiv 0$. Let $\alpha = (\lambda_1 + \lambda_2/2)r$. Let X_n be the total number of passengers waiting at both terminals at epoch T_n. $\{X_n\}$ is a sequence of independent identically distributed random variables with

$$P\{X_n = j\} = a_j(\alpha) = e^{-\alpha}\alpha^j/j!.$$

Let Z_j be the expected cost in the interval $(T_n, T_{n+1}]$ given $X_n = j$. Then, analogous to (1.7), we have (writing $g_3(k)$ to denote the dependence of the average cost for Problem 3 on the control limit value k)

$$g_3(k) = \frac{\sum_{j=0}^{\infty} a_j(\alpha) Z_j}{\sum_{j=0}^{\infty} a_j(\alpha) \cdot E(T_{n+1} - T_n | X_n = j)} . \tag{2.1}$$

From (1.3)-(1.4) we have:

$$E(T_{n+1} - T_n | X_n = j) = \begin{cases} r & \text{if } j \geq k \\ r + (k - j)/\lambda & \text{if } j < k. \end{cases}$$

Using a process analogous to $I_k(t|\underline{i})$ from Section I and omitting details:

If $j \geq k$, $Z_j = K + h(\frac{\alpha r}{2} + jr/2)$.

If $j < k$, $Z_j = K + h\left(j(k - j)/\lambda + (\frac{\alpha r}{2} + kr/2) + (k - j - 1)(k - j)/2\lambda\right)$.

Rewriting the kr/2 term as $jr/2 + (k - j)r/2$ and observing that

$\sum_{j=0}^{\infty} a_j(\alpha)hjr/2 = \frac{hr\alpha}{2}$,

$$g_3(k) = \frac{K + h\left(\alpha r + (1/2\lambda)\sum_{j=0}^{k-1} (k - j)(j + k + \lambda r - 1)a_j(\alpha)\right)}{r + (1/\lambda)\sum_{j=0}^{k-1} (k - j)a_j(\alpha)} \tag{2.2}$$

and

$$g_1(k) = g_3(k) - h\lambda_1 r/2. \tag{2.3}$$

We have made computations with (2.3) for various values of λ_1, λ_2, and K. We have set r = h = 1, and, in so doing, effectively have chosen the time scale so that r = 1 and cost units so that h = 1. We have computed $g_1(k)$ for k = 0, 1, 2, ..., 30 for 490 cases: 7 values of K, namely 0, 1, 2, 4, 8, 16, and 32; for each value of K, 7 values of λ and 7 of α(where $\alpha = (\lambda_1 + \lambda_2/2)r = \lambda_1 + \lambda_2/2$), namely .25, .5, 1, 2, 4, 8, and 16; and, for each value of λ and α, 5 values of λ_1. For each value of λ, the values of λ_1 are $.125\lambda$, $.25\lambda$, $.5\lambda$, $.75\lambda$, and $.875\lambda$; similarly, for each value of α, the values of λ_1 are $.125\alpha$, ..., $.875\alpha$. In each case, we find that $g(0) > g(1)$, implying that it is always a good idea to wait for at least one passenger, and that g(k) is convex in k on {1, 2, 3, ..., 30}. The minimizing value of k, call it k*, increases with λ and with K, and, as Table 1 indicates, to a first approximation k* is equal to $\sqrt{2\lambda K}$. (This approximation is suggested by Deb and Serfozo's result for a variant of Problem 3.) A complete table of results appears in the Appendix.

To see the effect of changing h, we note that doubling h would double the value of each cost unit and thus halve the value of K. To see the effect of changing r, we note that doubling r would double the size of each time unit and hence double λ_1 and λ_2; h, being the holding cost per customer per unit time, would also double, thus halving K. As a result, the rule of thumb that k* is equal to $\sqrt{2\lambda K}$ when r = h = 1 becomes k* is approximately equal to $\sqrt{2\lambda K/h}$ when r and h are unconstrained.

Table 1

VALUES OF k* AS A FUNCTION OF λ AND K

$\lambda_1 = \lambda_2 = \lambda/2$, K = 32			$\lambda_1 = \lambda_2 = \lambda/2$, λ = 2		
λ	$\sqrt{2\lambda K}$	k*	K	$\sqrt{2\lambda K}$	k*
.25	4.0	4	1	2.0	2
.50	5.6	6	2	2.8	2
1.00	8.0	8	4	4.0	4
2.00	11.2	11	8	5.6	5
4.00	16.0	15	16	8.0	8
8.00	22.4	21	32	11.2	11

III. A RULE WHEN THE QUEUE SIZE AT TERMINAL 2 IS NOT KNOWN

So, when the number of passengers waiting at each terminal is known to the controller, it is optimal to dispatch the shuttle if, and only if, the total number of passengers waiting at both terminals is at least k. We have not been able to derive the form of an optimal dispatch policy when the controller has knowledge only of the number of passengers waiting at terminal 1. Instead, we suggest a family of rules which are likely to be quite good. These rules are motivated by the form of the rule when complete passenger waiting information is available--that is, go if $i_1 + i_2 \geq k$. In the situation at hand, i_2 is not known, but, if the carrier has been waiting at terminal 1 for t minutes, he left terminal 2 empty $t + r/2$ minutes ago and, hence, the expected number of passengers waiting at terminal 2 is $\lambda_2(t + r/2)$, and we consider dispatch rules of the form: Go if $i_1 + \beta(t + r/2) \geq k$. Dropping the subscript, noting that k need not be integer, and defining $u = k - \beta r/2$, the rule can be expressed as: Go if $i + \beta t \geq u$. Equations (3.5)-(3.9), which are developed below, permit calculation of average cost per unit time, trip rate, and average number of passengers waiting as functions of the pair (u, β). Calculations made with these equations indicate that the optimal pair (u^*, β^*) is typically such that β^* is considerably larger than λ_2. For example, for $\lambda_1 = \lambda_2 = .5$ and $r = h = K = 1$, $(u^*, \beta^*) \cong (.7, .66)$, so β^* is about 30 percent larger than λ_2.

More important, these calculations coupled with those described in the previous section, suggest the value of knowing the number waiting at terminal 2. In this discussion, it seems more informative to make comparisons of the average number waiting rather than average cost. If an optimal SM rule for Problem 1 is used as a point of comparison, we find that the best of the (u, β) rules with approximately the same trip rate typically has the average number of passengers waiting 25 to 75 percent higher. For example, consider the parameters in the previous paragraph. The optimal SM rule for this case when both queues sizes are known to the controller is $k^* = 1$ and it yields .68 trips per unit time and an average of .42 passengers waiting. The (.7, .66) rule which is best

in the (u^*, β^*) family yields .67 trips per unit time and an average of .62 passengers waiting. The main effect of the lack of information is a 45 percent increase in the average number of passengers waiting.

To carry out our analysis, we again work with the $\{T_n\}$ and $\{X_n\}$ processes defined in Section II, except that X_n is now the number of passengers waiting at terminal 1 when the shuttle arrives there. Let D_j be the time the shuttle waits at terminal 1 when $X_n = j$. Clearly, $D_j = 0$ for $j \geq u$. For $j < u$

$$D_j = \left(T_{n+1} - (T_n + r) \,|\, X_n = j\right).$$

Let us begin by relating D_0 to passenger arrivals at terminal 1. It is helpful to start with Fig. 1:

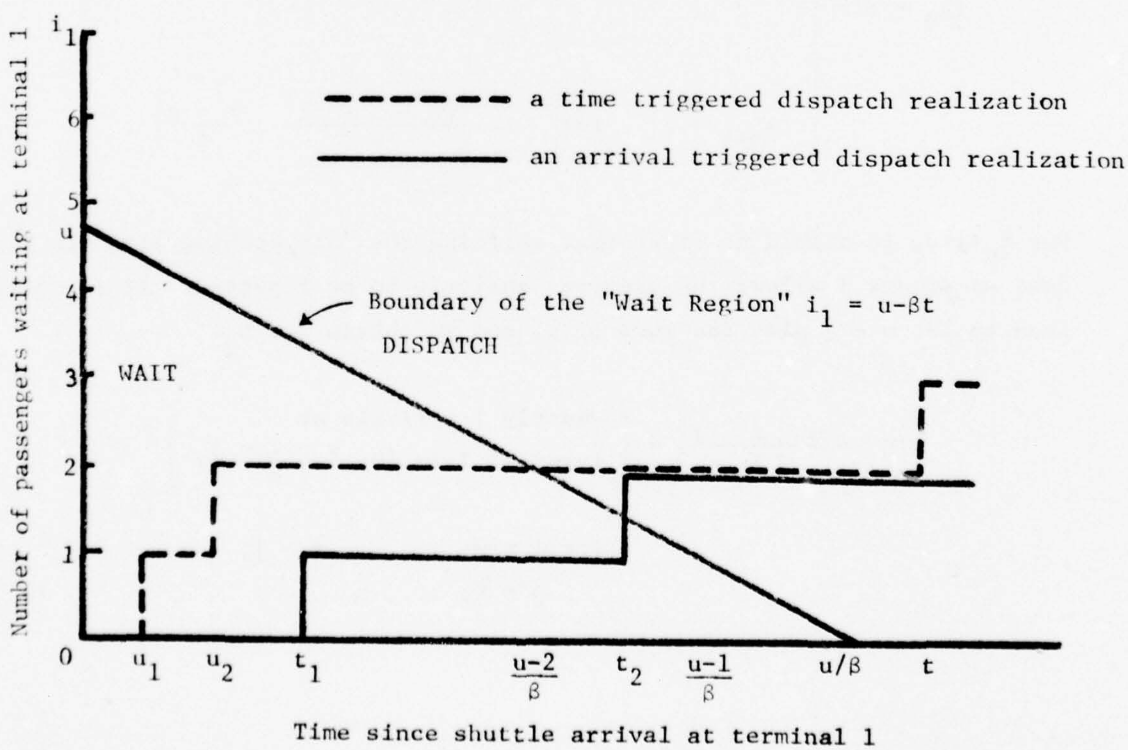

Fig. 1--Two realizations of the (u, β) dispatching mechanism

If arrivals to terminal 1 occur at t_1 and t_2 after the shuttle arrival at T_n, as illustrated by the solid line in Fig. 1, then $D_0 = t_2$, and we will say that the shuttle departure is "triggered by an arrival." If arrivals to terminal 1 occur at u_1, u_2, and u_3, as illustrated by the dotted line in Fig. 1, then $D_0 = (u - 2)/\beta$, and we will say that the shuttle departure is "triggered by time."

Let A_i be the time until i arrivals occur at terminal 1; A_i has a gamma distribution with parameters i and λ. From Fig. 1 we can see that, for time triggering to occur, there will be exactly i arrivals at terminal 1 for i = 0, 1, ..., m where m is the greatest integer less than u and the event $\{D_0 = \frac{u-i}{\beta}\}$ = the event {exactly i arrivals at terminal 1 in $[0, \frac{u-i}{\beta})$}.

For arrival triggering, we see that

$$\{D_0 = t\} = \begin{cases} \{A_i = t\} & \text{for} \quad t \varepsilon \left(\frac{u-1}{\beta}, \frac{u}{\beta}\right) \\ \{A_2 = t\} & \text{for} \quad t \varepsilon \left(\frac{u-2}{\beta}, \frac{u-1}{\beta}\right) \\ \cdot & \\ \cdot & \\ \cdot & \\ \{A_{m+1} = t\} & \text{for} \quad t \varepsilon \left(\frac{[u-m-1]^+}{\beta}, \frac{u-m}{\beta}\right) \end{cases} .$$

For $X_n = j < u$, it should be clear that shifting the "dispatching line" in Fig. 1 down an amount j allows the previous analysis to be repeated. It is sufficient then to let u - j play the role of u, and we obtain for D_j

$$\{D_j = \frac{u-i-j}{\beta}\} = \left\{ \begin{array}{l} \text{exactly } i \text{ arrivals at} \\ \text{terminal 1 in } [0, \frac{u-i-j}{\beta}) \end{array} \right\}$$

$$\text{for } i = 0, 1, \ldots, m - j; \quad j = 0, \ldots, m \qquad (3.1)$$

and

$$\{D_j = t\} = \{A_i = t\} \quad \text{for} \quad t\varepsilon \left(\frac{[u - i - j]^+}{\beta}, \frac{u - i - j + 1}{\beta}\right]$$

$$\text{for } i = 1, \ldots, m - j + 1; \quad j = 0, \ldots, m. \tag{3.2}$$

From (3.1) we get

$$P\left(D_j = \frac{u - i - j}{\beta}\right) = a_i\left(\lambda_1 \cdot \frac{u - i - j}{\beta}\right)$$

$$\text{for } i = 0, \ldots, m - j; \quad j = 0, \ldots, m. \tag{3.3}$$

From (3.2) we get

$$\frac{d}{dt} P\{D_j = t\} = \frac{\lambda_1^{\,i}\, t^{i-1}\, e^{-\lambda_1 t}}{(i - 1)!},$$

$$= \lambda_1 a_{i-1}(\lambda_1 t), \quad \text{for} \quad \frac{[u - i - j]^+}{\beta} < t \leq \frac{u - i - j + 1}{\beta}. \tag{3.4}$$

Let C_j^* be the expected cost incurred in the interval $(T_n, T_{n+1}]$ given $X_n = j$. Then

$$g = \sum_{j=0}^{\infty} a_j(\lambda_1 r)C_j^* \Big/ \sum_{j=0}^{\infty} a_j(\lambda_1 r)[ED_j + r]. \tag{3.5}$$

For the denominator, we have $ED_j = 0$ for $j > m$, and for $j \leq m$,

$$ED_j = \sum_{i=0}^{m-j} \frac{u - j - i}{\beta} \cdot a_i\left(\lambda_1 \cdot \frac{u - i - j}{\beta}\right)$$

$$+ \sum_{i=1}^{m-j+1} \int_{\frac{[u-i-j]^+}{\beta}}^{\frac{u-i-j+1}{\beta}} t \cdot \lambda_1 a_{i-1}(\lambda_1 t)dt$$

$$= \sum_{i=1}^{m-j+1} \frac{1}{\lambda_1} \left\{ \sum_{k=0}^{i} \left[a_k\left(\lambda_1 \frac{[u - i - j]^+}{\beta}\right) - a_k\left(\lambda_1 \frac{u - i - j + 1}{\beta}\right)\right] + a_i\left(\lambda_1 \cdot \frac{u - i - j + 1}{\beta}\right)\right\} \tag{3.6}$$

where, in order to evaluate the integral, we note that the integrand can be rewritten as $a_i(\lambda_1 t) \cdot i$ and use the identity

$$\int_v^w a_n(\lambda x)dx = \frac{1}{\lambda} \sum_{k=0}^{n} [a_k(\lambda v) - a_k(\lambda w)]. \tag{3.7}$$

For the numerator, we have, omitting the details, for $j > m$

$$C_j^* = K + h\lambda r^2/2. \tag{3.8}$$

For $j \leq m$,

$$C_j^* = A + B + C \tag{3.9}$$

where $A = \lambda h r^2/2 + K$,

$$B = h \cdot \sum_{i=0}^{m-j} a_i\left(\lambda_1 \frac{(u-j-i)}{\beta}\right) \cdot \begin{bmatrix} r\lambda_2\left(\frac{u-j-i}{\beta}\right) \\ + \lambda_2\left(\frac{u-j-i}{\beta}\right)^2/2 \\ + j \cdot \left(\frac{u-j-i}{\beta}\right) \\ + i \cdot \left(\frac{u-j-i}{\beta}\right)/2 \end{bmatrix}$$

and

$$C = h \cdot \sum_{i=1}^{m-j+1} \int_{\frac{[u-j-i]^+}{\beta}}^{\frac{u-j-i+1}{\beta}} \lambda_1 a_{i-1}(\lambda_1 t) \begin{bmatrix} r\lambda_2 t \\ + \lambda_2 t^2/2 \\ + jt \\ + i\,t/2 \end{bmatrix}$$

$$= h \cdot \sum_{i=1}^{m-j+1} (r\lambda_2 + j + i/2)\,\frac{i}{\lambda_1} \sum_{k=0}^{i} \begin{bmatrix} a_k\left(\lambda_1 \cdot \frac{[u-j-i]^+}{\beta}\right) \\ - a_k\left(\lambda_1 \cdot \frac{u-j-i+1}{\beta}\right) \end{bmatrix}$$

$$+ h \cdot \sum_{i=1}^{m-j+1} \left(\lambda_2 i(i+1)/2\lambda_1^2\right) \sum_{k=0}^{i+1} \begin{bmatrix} a_k\left(\lambda_1 \cdot \frac{[u-j-i]^+}{\beta}\right) \\ - a_k\left(\lambda_1 \cdot \frac{u-j-i+1}{\beta}\right) \end{bmatrix} .$$

A gives expected cost incurred after the shuttle leaves terminal 1, excluding costs associated with arrivals occurring during the shuttle's wait. B gives expected cost associated with the shuttle's wait, given time triggering. The first of the four terms multiplied by a_i gives cost incurred from (1) those passengers waiting at terminal 2 when the shuttle arrives at terminal 1, incurred during the shuttle's stay at terminal 1, and (2) those passengers arriving at terminal 2 during the shuttle's stay at terminal 1, incurred during the shuttle's trip to terminal 2. The second term gives cost from arrivals to terminal 2 during the stay, incurred during the stay. The third term gives cost from those waiting at terminal 1 when the shuttle arrives there, incurred during the stay. The fourth term gives cost from arrivals at terminal 1 during the stay, incurred during the stay. C gives expected cost associated with the shuttle's stay, given arrival triggering.

In writing down A, B, and C we have used a process like $I_k(t|\underline{i})$ from Section I, and have used (3.7) to simplify C.

IV. STOCHASTIC TRAVEL TIMES

For random variable travel times, we have results analogous to the constant travel time case when both queue sizes are known. This extension is outlined briefly below. When dispatching is done without knowledge of the situation at terminal 2, a broader class of rules than those investigated in the previous section seems appropriate when travel times are stochastic. For example: suppose the previous round trip travel time becomes known when the shuttle arrives at terminal 1. The longer it is, the larger the number of passengers likely to be waiting now at terminal 2, and it would be natural to include this conditional expectation in a rule for deciding when to go.

Let D and B be travel times from terminal 1 to terminal 2 and from 2 to 1, respectively, and let $R = D + B$. We assume that successive values of D are independent, with distribution function F, that successive values of B are independent, with distribution function G, and that the D's and B's are independent of each other and of the arrival and dispatching processes. Then, the development parallels that of Section I, with the following modifications.

For Problem 1, C^W, p^W and τ^W are unchanged and C^G, p^G and τ^G are given below.

We have
$$\tau^G = ER. \tag{4.1}$$

To get C^G, we look at the random variable $I_k(t|\underline{i})$ defined in Section I. Let $W_1(t)$ be the expected waiting cost at terminal 1 conditioned on $R = t$. Then,

$$W_1(t) = h \int_o^t EI_1(y|\underline{i})dy$$
$$= h\lambda_1 t^2/2.$$

$$p^G_{ij} = \sum_{j_1+j_2=j} p^G_{\underline{i}\underline{j}}$$

where the $p^G_{\underline{i}\underline{j}}$ are given in (4.3) does not alter their validity.

It can be shown that this replacement converts their problem into one that is equivalent (in the sense described in Section I and in the Appendix) to Problem 2 with stochastic travel times. Given the equivalence of Problems 1 and 2 in this stochastic travel time case, we can then conclude that a control limit rule is optimal for Problem 1.

Letting H be the distribution function of R, the unconditional expected waiting cost at terminal 1 is

$$W_1 = \int_o^\infty (h\lambda_1 t^2/2)dH(t)$$

$$= h\lambda_1 ER^2/2.$$

Using similar analysis for terminal 2, we obtain

$$C^G(\underline{i}) = K + hi_2 ED + \frac{h}{2}(\lambda_1 ER^2 + \lambda_2 EB^2). \tag{4.2}$$

Since arrivals in D and in B are independent, we can write

$$p^G_{\underline{i}\underline{j}} = \sum_{k=0}^{j_1} \int_o^\infty a_{j_1-k}(\lambda_1 t)dD(t) \cdot \int_o^\infty a_k(\lambda_1 t)\ a_{j_2}(\lambda_2 t)dB(t). \tag{4.3}$$

For Problem 2, we can add $hi_1 ED$ to $C^G(\underline{i})$, and, analogously to Section I, show that $g_2 = g_1 + h\lambda_1 ED$.

For Problem 3, the transformations given in Section I are appropriate for C^W, p^W and τ^W. For τ^G, use (4.1). For C^G, use

$$C^G(i) = \left[K + (h/2)(\lambda_1 ER^2 + \lambda_2 EB^2)\right] + i \cdot hED. \tag{4.4}$$

And for p^G use

$$p^G_{ij} = \int_o^\infty a_j(\lambda t)dH(t). \tag{4.5}$$

Deb and Serfozo proved that a rule of the form dispatch the carrier if, and only if, $i > k^*$ is optimal for the problem specified by (1.3), (1.12), (1.13), (4.1), (4.4), and (4.5). Examination of their proofs reveals that replacing (4.5) by

V. AN ALTERNATIVE CRITERION

An alternative to our approach of assigning costs both to passenger waiting and to trips made by the carrier is to view the minimization of passenger waiting as our objective while constraining the average number of trips made by the carrier to not exceed a preassigned limit. For any rule A, let $T(A)$ be the expected cycle time--that is, the expected time between successive arrivals of the carrier at terminal 1. Let $W(A)$ be the expected total waiting cost in a cycle. Define $Q(A)$ as $W(A)/T(A)$. Formally, then, our alternative to Problem 1 is to choose that SM rule A' that minimizes $Q(A)$ over the set of all SM rules that satisfy $T(A) \geq Z$. Call this Problem 4.

From (1.7) we have

$$g_1(A) = [K + W(A)]/T(A) = K/T(A) + Q(A).$$

This suggests that it should be possible to treat Problem 4 directly and show that a properly chosen randomization among control limit rules is optimal.* For brevity, we will simply show that, for properly chosen values of Z, a control limit rule is optimal. Consider any fixed value of K. We have shown that among the SM rules that minimize g_1 is (at least) one of the form: Dispatch if, and only if, $i_1 + i_2 > k$. Let * be any such optimal rule. If $Z = T(*)$, rule * is optimal (among the SM rules) for Problem 4. To see this, consider any SM rule A for which $T(A) \geq Z$. Then, we obtain $Q(A) \geq Q(*)$ by noting that: $g_1(A) \geq g_1(*)$ or $Q(A) + K/T(A) \geq Q(*) + K/T(*)$ or $Q(A) - Q(*) \geq K(1/T(*) - 1/T(A)) \geq 0$, where the last inequality results from $T(A) \geq Z = T(*)$.

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*For an example of such analysis in an equipment replacement context, see Kolesar [4].

APPENDIX

AN ALTERNATE APPROACH TO THEOREM 1

We show that an equivalence between Problems 2 and 3 allows us to claim that there is, among the optimal SM rules for Problem 2, one which is a function only of $i_1 + i_2$.

We claim that, for any SM rule A for Problem 2 (our system), there exists an SM rule B for Problem 3, with (1.15) replacing (1.14) as the definition of p^G_{ij}, (the one-dimensional system) with the following property: Imagine an observer who is informed

(a) each time an arrival occurs while the shuttle is waiting (but not where it occurs for our system),

(b) each time the shuttle returns to terminal 1, being told the total number of passengers now waiting, and

(c) each time the shuttle is dispatched.

The observer will not be able to identify which process is the one-dimensional system, controlled by B, and which is ours, controlled by A. To see why, consider the following example. Let A(0, 0) = 0, A(1, 0) = 1/3, A(0, 1) = 4/5, and $A(\underline{i}) = 1$ for $i_1 + i_2 \geq 2$. (In words, policy A is: wait if no one is waiting; go if two or more are waiting (anywhere); go with probability 1/3 if one is waiting at terminal 1 and no one at 2; go with probability 4/5 if one is waiting at terminal 2 and no one at 1.) Then, the probability that we will dispatch when there is exactly one person waiting (somewhere) is

$$B(1) = \left[p_0(\lambda_1 r)p_0(\lambda_2 r/2)\left(\frac{\lambda_1}{\lambda} \cdot \frac{1}{3} + \frac{\lambda_2}{\lambda} \cdot \frac{4}{5}\right) + p_1(\lambda_1 r)p_0(\lambda_2 r/2) \cdot \frac{1}{3} + p_0(\lambda_1 r)p_1(\lambda_2 r/2) \cdot \frac{4}{5}\right] \Big/ \left[p_0(\lambda_1 r)p_0(\lambda_2 r/2) + p_1(\lambda_1 r)p_0(\lambda_2 r/2) + p_0(\lambda_1 r)p_1(\lambda_2 r/2)\right].$$

So the observer would not be able to tell our system from the one-dimensional one that used that value of $B(1)$ (and $B(0) = 0$; $B(k) = 1$, $k \geq 2$).

It is easier to see that there is an equivalent B for a given A than it is to specify it. Speaking roughly, we do so by:

$$B(k) = \sum_{j=0}^{k} P\begin{pmatrix}\text{dispatch from state } (j,\ k-j) \text{ if} \\ \text{reach } (j,\ k-j)\text{, under rule A}\end{pmatrix} \cdot$$

$$P\left(\begin{array}{l|l}\text{reach state } (j,\ k-j) & \text{reach } k \text{ passengers} \\ \text{under rule A} & \text{waiting in total} \\ & \text{under rule A}\end{array}\right).$$

Formally: Let

$$R(j,\ k-j) = P\begin{pmatrix}\text{under rule A, system enters} \\ \text{state } (j,\ k-j) \text{ before shuttle} \\ \text{is next dispatched, given that} \\ \text{it has just been dispatched}\end{pmatrix}$$

and let

$$\beta(j,\ k-j) = R(j,\ k-j) \Big/ \sum_{i=0}^{k} R(i,\ k-i)$$

$$\text{for } 0 \leq j \leq k,\quad k = 0, 1, \ldots .$$

(Explicit calculation of $\beta(j,\ k-j)$ is possible but sloppy; it depends on A; but mainly it is irrelevant to our purpose here.) Since a one-step transition to any state is possible if we dispatch the shuttle, $\beta(j,\ k-j) > 0$ for all $0 \leq j \leq k$. By definition, $\sum_{j=0}^{k} \beta(j,\ k-j) = 1$ for $k = 0, 1, \ldots$. The equivalent B for the given A is

$$B(k) = \sum_{j=0}^{k} A(j,\ k-j) \cdot \beta(j,\ k-j) \qquad k = 0, 1, \ldots . \tag{A.1}$$

Now, by construction of the cost functions, if B is equivalent to A, we have the same average cost per unit time for our system and the one-dimensional one. But, to minimize cost for the one-dimensional system, Deb and Serfozo have proved that among optimal rules (including non-SM rules) is an SM rule, call it B*, such that

$$B^*(k) = 0 \quad \text{for} \quad k < k^*; \quad B^*(k) = 1 \quad \text{otherwise.} \tag{A.2}$$

(Actually, they treat a class of processes that include (1.3), (1.4), and (1.11)-(1.14) as a special case. An examination of the proofs of their results leading to (A.2) reveals that using (1.15) instead of (1.14) does not alter them.)

We claim that there is an optimal SM rule for the system (1.1)-(1.5) and (1.9) which satisfies both (A.1) and (A.2). Any rule A satisfying (A.1) alone is equivalent to a B which is no better than B*, and we now show that there is an A* equivalent to B*. It is $A^*(j, k - j) = B^*(k)$ for $0 \leq j \leq k$, $k = 0, 1, \ldots$. And, to return to Problem 1, we have from (1.8) that an optimal rule for (1.1)-(1.5) and (1.9) is optimal for (1.1)-(1.6). That is, we have proved Theorem 1.

OPTIMAL VALUES OF THE CONTROL LIMIT FOR PROBLEM 1

The entries in the following table are values of k* such that, for the indicated cost and arrival parameters, an optimal SM rule is to dispatch the carrier if, and only if, a total of at least k* passengers are waiting at the two terminals.

Table A.1

OPTIMAL DISPATCHING NUMBERS (k^*) FOR AN INFINITE CAPACITY SHUTTLE OPERATING WITH COMPLETE PASSENGER WAITING INFORMATION
($r = h = 1$)

	Values of λ_1	Values of K (trip cost) → 0	1	2	4	8	16	32
$\lambda = 0.25$	0.03125	1	1	1	1	2	3	4
	0.06250	1	1	1	1	2	3	4
	0.12500	1	1	1	1	2	3	4
	0.18750	1	1	1	1	2	3	4
	0.21875	1	1	1	1	2	3	4
$\alpha/r = 0.25$	0.03125	1	1	1	2	3	4	5
	0.06250	1	1	1	2	3	4	5
	0.12500	1	1	1	2	2	3	5
	0.18750	1	1	1	2	2	3	4
	0.21875	1	1	1	2	2	3	4
$\lambda = 0.5$	0.06250	1	1	1	2	3	4	5
	0.12500	1	1	1	2	3	4	6
	0.25000	1	1	1	2	3	4	6
	0.37500	1	1	1	2	3	4	6
	0.43750	1	1	1	2	3	4	6
$\alpha/r = 0.5$	0.06250	1	1	2	2	4	5	7
	0.12500	1	1	2	2	3	5	7
	0.25000	1	1	2	2	3	5	7
	0.37500	1	1	2	2	3	4	6
	0.43750	1	1	2	2	3	4	6

	Values of λ_1	Values of K (trip cost) → 0	1	2	4	8	16	32
$\lambda = 1.0$	0.125	1	1	2	3	4	5	8
	0.250	1	1	2	3	4	5	8
	0.500	1	1	2	3	4	5	8
	0.750	1	1	2	3	4	6	8
	0.875	1	2	2	3	4	6	8
$\alpha/r = 1.0$	0.125	1	1	2	3	5	7	10
	0.250	1	1	2	3	5	7	10
	0.500	1	2	2	3	4	6	9
	0.750	1	2	2	3	4	6	9
	0.875	1	2	2	3	4	6	8
$\lambda = 2.0$	0.250	1	1	2	3	5	7	10
	0.500	1	2	2	3	5	7	11
	1.000	1	2	2	4	5	8	11
	1.500	1	2	3	4	5	8	11
	1.750	1	2	3	4	6	8	11
$\alpha/r = 2.0$	0.250	1	2	2	4	6	9	14
	0.500	1	2	3	4	6	9	14
	1.000	1	2	3	4	6	9	13
	1.500	1	2	3	4	6	8	12
	1.750	1	2	3	4	6	8	12

	Values of λ_1	Values of K (trip cost) → 0	1	2	4	8	16	32
$\lambda = 4.0$	0.5	1	2	3	4	6	10	14
	1.0	1	2	3	4	7	10	15
	2.0	1	2	3	5	7	10	15
	3.0	2	3	4	5	8	11	16
	3.5	2	3	4	5	8	11	16
$\alpha/r = 4.0$	0.5	1	2	3	5	8	12	18
	1.0	1	2	3	5	8	12	18
	2.0	1	2	3	5	8	12	18
	3.0	2	3	4	5	8	12	17
	3.5	2	3	4	6	8	12	16
$\lambda = 8.0$	1.0	1	2	3	5	8	13	19
	2.0	1	2	3	5	8	13	20
	4.0	2	3	4	6	9	14	21
	6.0	3	4	5	7	10	15	22
	7.0	4	5	6	8	11	16	22
$\alpha/r = 8.0$	1.0	1	2	3	5	9	15	24
	2.0	1	2	3	5	9	15	24
	4.0	2	4	4	6	10	16	24
	6.0	3	4	5	7	11	16	23
	7.0	4	5	6	8	11	16	23

	Values of λ_1	Values of K (trip cost) → 0	1	2	4	8	16	32
λ = 16.0	2.0	1	3	3	5	9	16	25
	4.0	3	3	4	6	10	17	26
	8.0	4	5	6	8	12	19	28
	12.0	7	7	8	10	14	21	30+
	14.0	7	8	9	11	15	22	30+
α/r = 16.0	2.0	3	5	5	6	9	17	30+
	4.0	4	4	4	7	10	18	30+
		5	6	6	8	12	20	30+
		7	8	8	10	14	21	30+
		8	9	9	11	15	22	30+

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